COMP 232 Mathematics for Computer Science
Fall 2012
Midterm Exam

Name: ________________________ Total Points:

ID: ________________________ ________ / 71

Instructions. This is a closed book exam. The only allowed tool is an ENCS approved calculator. Provide all answers in this booklet. Use pen, not pencil. Do not detach any pages from this exam!

($2^{\text{pts}}_{\text{ea.}}$) 1. Let the universe of discourse be $\mathbb{Z}^+$, the set of positive integers. For each of the following sentences, indicate whether it is true or false. You get +2 points for each correct answer, -2 points for each wrong answer, and 0 points for "don't know." However, the total for this question will not be less that 0. [ ] 10 pts

(a) $\forall x\big((x < 0) \vee (x \leq 2x)\big)$

☐ False ☑ True ☐ Don't know!

(b) $\exists x \exists y\big((x + y = 0) \vee (x \cdot y = 0)\big)$

☐ True ☑ False ☐ Don't know!

(c) $\forall x \forall y(x \cdot y \geq x + y)$

☑ False ☐ True ☐ Don't know!

(d) $\exists x \exists y\big((x = 3) \vee (y = 4)\big)$

☑ True ☐ False ☐ Don't know!

(e) $\exists x \forall y \exists z\big((y = x + z) \wedge (z \leq x)\big)$

☐ True ☑ False ☐ Don't know!

[ ] 10 pts

($6^{\text{pts}}_{\text{ea.}}$) 2. Here you are to prove propositional equivalences using the laws at the last page of this exam

12 pts

(a) Here is a proof that $p \to (q \to r) \equiv (p \wedge q)$.

Step			Law applied
$p \to (q \to r)$	$\equiv$	$\neg p \vee (q \to r)$	Implication
	$\equiv$	$\neg p \vee (\neg q \vee r)$	Implication
	$\equiv$	$(\neg p \vee \neg q) \vee r$	Associativity
	$\equiv$	$\neg(p \wedge q) \vee r$	de Morgan
	$\equiv$	$(p \wedge q) \to r$	Implication

In the rightmost column above, fill in the law applied for each step (see last page of this booklet for a list of laws)

(b) In the table below, construct a proof of the equivalence

$$(r \vee p) \to (r \vee q) \equiv r \vee (p \to q)$$

similarly to (a).

Step			Law applied
$(r \vee p) \to (r \vee q)$	$\equiv$	$\neg(r \vee p) \vee (r \vee q)$	Implication
	$\equiv$	$\Big((\neg r) \wedge (\neg p)\Big) \vee (r \vee q)$	de Morgan
	$\equiv$	$\Big(\Big((\neg r) \wedge (\neg p)\Big) \vee r\Big) \vee q$	Associativity
	$\equiv$	$\Big(r \vee \Big((\neg r) \wedge (\neg p)\Big)\Big) \vee q$	Commutativity
	$\equiv$	$\Big(\Big(r \vee (\neg r)\Big) \wedge \Big(r \vee (\neg p)\Big)\Big) \vee q$	Distributivity
	$\equiv$	$\Big(\Big(r \vee (\neg p)\Big) \wedge \Big(r \vee (\neg r)\Big)\Big) \vee q$	Commutativity
	$\equiv$	$\Big(\Big(r \vee (\neg p)\Big) \wedge T\Big) \vee q$	Excluded middle
	$\equiv$	$\Big(r \vee (\neg p)\Big) \vee q$	Identity
	$\equiv$	$r \vee \Big((\neg p) \vee q\Big)$	Associativity
	$\equiv$	$r \vee (p \to q)$	Implication

12 pts

(3[pts ea.]) 3. We know that $\{\wedge, \neg\}$ forms a functionally complete set of operators, meaning that any other operator can be defined in terms of $\{\wedge, \neg\}$ only, for example

9 pts

$$\begin{aligned} p \vee q &=_{\mathsf{def}} \neg(\neg p \wedge \neg q) \\ p \rightarrow q &=_{\mathsf{def}} \neg(p \wedge \neg q) \end{aligned}$$

The Shaffer stroke $\uparrow$ is a binary operator that has the following truth table:

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Show that the Shaffer stroke by itself is functionally complete, by defining in the space below, the following operators by <u>using the Shaffer stroke only</u>:

(a) $\neg p =_{\mathsf{def}} p \uparrow p$

(b) $p \wedge q =_{\mathsf{def}} (p \uparrow q) \uparrow (p \uparrow q)$

(c) $p \vee q =_{\mathsf{def}} (p \uparrow p) \uparrow (q \uparrow q)$

(6[pts]) 4. The negation of the statement $\forall x \neg \forall y \exists z (P(x, z) \wedge Q(z, y))$ is

6 pts

- [x] $\exists x \forall y \exists z (P(x, z) \wedge Q(z, y))$
- [ ] $\forall x \exists y \forall z (\neg P(x, z) \vee \neg Q(z, y))$
- [ ] $\forall x \forall y \exists z (\neg P(x, z) \wedge \neg Q(z, y))$
- [ ] $\forall x \exists y \forall z (\neg P(x, z) \wedge \neg Q(z, y))$
- [ ] $\exists x \exists y \forall z (P(x, z) \vee \neg Q(z, y))$

(6[pts]) 5. Which of the following statements is the contrapositive of the statement "You win the game if you know the rules but are not overconfident."

6 pts

- [x] "If you lose the game then you don't know the rules or you are overconfident."
- [ ] "If you don't know the rules or are overconfident then you lose the game."
- [ ] "If you don't know the rules and are overconfident then you win the game."
- [ ] "A sufficient condition that you win the game is that you know the rules or you are not overconfident."
- [ ] "A necessary condition that you know the rules or you are not overconfident is that you win the game."

(6^{pts}) 6. To prove $p \wedge (\neg q) \Rightarrow r \vee (\neg s)$ by contradiction, which of the following propositions is the appropriate one to prove.

6 pts

- ☐ $\big((\neg p) \wedge q \wedge s \wedge (\neg r)\big) \Rightarrow False$
- ☐ $\big((\neg p) \wedge q \wedge r \wedge (\neg s)\big) \Rightarrow False$
- ☑ $\big((\neg q) \wedge p \wedge (\neg r) \wedge s\big) \Rightarrow False$
- ☑ $\big((\neg q) \wedge p \wedge s \wedge (\neg r)\big) \Rightarrow False$
- ☑ $\big(p \wedge (\neg q) \wedge (\neg r) \wedge s\big) \Rightarrow False$

(8^{pts}) 7. $|\mathcal{P}\big((A \times B) \cup (B \times A)\big)| = |\mathcal{P}\big((A \times B) \cup (A \times B)\big)|$ <u>if and only if</u>

8 pts

- ☐ $A = \emptyset$ or $B = \emptyset$ or $A \cap B = \emptyset$
- ☐ $A = B$
- ☐ $A = \emptyset$ or $B = \emptyset$
- ☐ $B = \emptyset$ or $A = B$
- ☑ $A = \emptyset$ or $B = \emptyset$ or $A = B$

($2^{\text{pts}}_{\text{ea.}}$) 8. Consider the following proof that there is no least (smallest) positive real number.

6 pts

Proof: Suppose to the contrary that there is a real number x, such that x is positive and ____(a)____ for all positive real numbers y. Consider the number $x/2$. Then ____(b)____ because x is positive. Hence ____(c)____, which is a contradiction.■

In (a), (b), and (c) below, fill in the corresponding blanks in the proof, such that you get a valid proof.

(a) $x \leq y$

(b) $x/2 > 0$

(c) $x \leq x/2$

20 pts

($2^{\text{pts}}_{\text{ea.}}$) 9. The symmetric difference between sets A and B is defined as

8 pts

$$A \oplus B = (A - B) \cup (B - A).$$

For each of the proposed identity involving $\oplus$ below, state whether the identity is true or false. You get +2 points for each correct answer, -2 points for each wrong answer, and 0 points for "don't know." However, the total for this question will not be less that 0.

(a) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

☐ False ☑ True ☐ Don't know!

(b) $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$.

☑ True ☐ False ☐ Don't know!

(c) $((A \oplus B) \oplus C) \cup (A \cap B \cap C) = A \cup B \cup C$.

☑ False ☐ True ☐ Don't know!

(d) $(A \oplus B) \cup B = A \cup B$

☑ True ☐ False ☐ Don't know!

. — End of Exam — .

8 pts

Basic logical equivalences:

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotency
$\neg(\neg p) \equiv p$	Double negation
$p \vee (\neg p) \equiv T$	Law of excluded middle
$p \wedge (\neg p) \equiv F$	Law of contradiction
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutativity
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associativity
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributivity
$p \wedge (p \vee q) \equiv p$ $p \vee (p \wedge q) \equiv p$	Absorption laws
$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$ $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$	de Morgan's laws

0 pts